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THE MATHEMATICS STUDENT

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ON EXPRESSING THE EQUATIONS OF ANY CONGRUENCE OF CURVES IN THE HAMILTONIAN FORM.

BY K. NAGABHUSHANAM,
Andhra University, Waltair.

1. Let the equations of a given congruence of curves in an n dimensional manifold be

$$x^r = c_r \quad (r = 1, 2, \dots n-1); \quad x^n = \text{anything} \quad \dots \quad (A)$$

where $c_1, c_2, \dots c_{n-1}$ are arbitrary constants.

We inquire if the equations (A) can be expressed in the Hamiltonian form

$$\left.\begin{aligned} dq^\alpha - \frac{\partial H}{\partial p_\alpha}\, dt &= 0 \\ -\,dp_\alpha - \frac{\partial H}{\partial q^\alpha}\, dt &= 0 \end{aligned}\right\} (\alpha = 1, 2, \dots r) \dots \quad (B)$$

2. Suppose that (A) can be put in the form (B) by a change of variables. On changing the variables in (B) to $x^1, x^2, \dots x^p$ the equations (B) take the form*

$$a_{ik}\, dx^k = 0, \qquad (i = 1, 2, \dots n) \qquad \dots \quad (C)$$

where $a_{ik} = \dfrac{\partial X_i}{\partial x^k} - \dfrac{\partial X_k}{\partial x^i}$, and the repeated index stands for summation from 1 to n. The problem then reduces to finding the covariant vector (X_i) such that the Pfaff's First System of equations (C) of the form

* See E. T. Whittaker: *Analytical Dynamics*, 3rd ed., p. 308.

$X_i\, dx^i$ is identical with (A). The co-ordinate system $(x^1, x^2 \ldots x^n)$ is used throughout the discussion. We shall treat the two cases when n is odd and even separately.

2. (*a*) $n = 2r + 1.$

Suppose that it is possible to write (A) in the form (C). Substituting $dx^1 = dx^2 = \ldots dx^{n-1} = 0$ in (C), we have

$$a_{in}\, dx^n = 0, \qquad (i = 1, 2, \ldots n).$$

As $dx^n \neq 0$ along the curves of the congruence,

$$a_{in} = 0, \qquad (i = 1, 2, \ldots n). \qquad \ldots \quad \text{(D)}$$

Also the number of independent equations of the congruence is $n - 1\ (= 2r)$, so that the rank of the matrix $\| a_{ik} \|$ is exactly $2r$. Whenever (X_i) satisfies the two conditions

(i) $a_{in} = 0, \qquad (i = 1, 2 \ldots n).$

and (ii) the rank of $\| a_{ik} \|$ is $2r$, the equations (C) yield (A) as the very conditions of consistency.

(*b*) $n = 2r + 2$

The set of equations got by leaving out the first equation in (A) can be brought under the form (C), as seen in (*a*). In this case, the $2r$ independent equations in (C) must be supplemented by the equation $x^1 =$ constant

In both cases the $2r$ equations in the form (C) can be written* in the form (B) by a suitable change of the variables to $q^1, \ldots q^r, p_1, \ldots p_r$. We may state the result as follows:

The equations (A) *are equivalent to a system of Hamiltonian equations when n is odd and to a set of Hamiltonian equations together with an equation of the type $\phi =$ constant (ϕ being an integral of the congruence) when n is even.*

---

* In the next section we shall show how to construct the form $X_i dx_i$. When it is expressed as $\sum_{\alpha=1}^{r} p_\alpha\, dq^\alpha - H\, dt$, the two forms (B) and (C) become identical.

3. Lastly, we shall determine the form of (X_i). From (D) we have

$$\frac{\partial X_j}{\partial x^{2r+1}} = \frac{\partial X_{2r+1}}{\partial x^j}, \qquad (j = 1, 2 \ldots 2r).$$

If we write

$$X_j = \frac{\partial \psi}{\partial x_j} + f_j (x^1, x^2, \ldots x^{2r})$$

and

$$X_{2r+1} = \frac{\partial \psi}{\partial x^{2r+1}} + f_{2r+1} (x^{2r+1})$$

where ψ is an arbitrary function of all the variables, then

$$X_i dx^i = \frac{\partial \psi}{\partial x^i} dx^i + f_i dx^i,$$

i.e.,
$$(X_i) = \text{grad } \psi + (f_i) \qquad \ldots \quad \text{(E)}$$

Since f_{2r+1} is a function of x^{2r+1} only, we may write

$$(f_i) = \text{grad } \phi + (f_1, f_2, \ldots f_{2r}, 0)$$

where $\phi = \int f_{2r+1} dx^{2r+1}$.

Now (E) becomes

$$(X_i) = \text{grad } (\psi + \phi) + (f_1, f_2, \ldots f_{2r}, 0) \qquad \ldots \quad \text{(F)}$$

The equation contains in itself the condition (D). The second condition that the rank of $\| a_{ik} \|$ is $2r$ is equivalent to the statement that the rank or class* of the form $\omega_{2r} = \sum_{j=1}^{2r} f_j dx^j$ is $2r$. Also any of the functions $f_1, f_2, \ldots f_{2r}$ is either a constant or an integral of the congruence.

I hereby thank Dr. R. Vaithianathaswamy for his kind help.

---

* The rank cannot be greater than $2r$, for ω_{2r} is expressed in $2r$ variables. It cannot be less than $2r$ which is the rank of $\| a_{ik} \|$.

THEOREMS CONCERNING TWO HOMOGRAPHIC TETRADS OF POINTS ON A CONIC.*

By T. R. Raghava Sastri,
Loyola College, Madras.

Two homographic ranges on a conic S are uniquely determined by two triads of corresponding points (A, B, C) and (A′B′C′) on it. If D, D′ are a pair of corresponding points, the corresponding cross joins like AD′, A′D meet on the homographic axis and the joins of corresponding points like AA′ envelop a conic having double contact with S at the double points.

The purpose of this paper is to draw attention to certain results relating to two such tetrads.

Theorem. If (ABCD) aad (A′B′C′D′) are two homographic tetrads of points on a conic S, the six meets of pairs of corresponding joins like AB, A′B′ together with the six meets of pairs of non-corresponding joins like AB, C′D′ lie on a conic R passing through the pole P with respect to S of the homographic axis of the two tetrads ... (A)

Let BC, B′C′ meet in X and AD, A′D′ in X′ with Y, Y′, Z, Z′ denoting similar meets. Let BC and A′D′ meet in L, B′C′ and AD in L′, with M, M′, N, N′ denoting similar meets. Also let the homographic axis meet the conic S in H and H′.

We start with the fundamental fact that

PX and YZ, PY and ZX and PZ and XY are conjugate pairs for S ... (1.1)

This follows from the Pascal hexagon CABC′A′B′ wherein the Pascal line YZ passes through the meet of BC′ and B′C which is evidently the pole of PX for S. Similarly it can be proved that the other pairs are also conjugate for the conic.

* I am indebted to Mr. V. Ramaswamy Aiyar for helpful suggestions, one of them being the enunciation of the theorem (A) in the present form.

Hence the polar of P (*i.e.*, the homographic axis) cuts the opposite pairs of sides of the quadrangle PXYZ in involution with H, H′ as double points. Hence also the pencil of conics through PXYZ will determine the same involution on HH′.

By taking the two corresponding triads B, C, D and B′, C′, D′, it similarly follows that the pencil of conics through PXY′Z′ determines the same involution on HH′. Evidently therefore

the conic PXYY′Z must pass through Z′ as well ; and by parity of reasoning through X′ also ... (1·2)

Again, from (1.1), PX, YZ as well as PX, Y′Z′ are conjugate pairs for the conic S. Therefore YZ and Y′Z′ must intersect in the pole of PX which is necessarily on the polar of P, *i.e*, the homographic axis. Similarly ZX and Z′X′, XY and X′Y′ must intersect on this line. But from (1.2) the six points lie on a conic R. Hence from the hexagon XYZX′Y′Z′,

the line HH′ must contain the meet of ZX′ and Z′X ; whence XX′ and ZZ′, similarly YY′ also, pass through the pole of HH′ with respect to the conic R ... (1.3)

If HH′ meets R in I and I′, it follows therefore that each pair X, X′ ; Y, Y′ ; Z, Z′ divide I, I′ harmonically. As P is a point on R, P (XX′II′) = — 1. Hence their poles for S form a harmonic range on HH′. But the pole of PX is the meet of BC′ with HH′ and the pole of PX′ is the meet of AD′ with HH′. Also the poles of PI and PI′ are I′ and I respectively. The pencil of conics through ABC′D′, of which S is one, cuts HH′ in involution ; but as (HH′, II′) and the above range are harmonic, therefore I, I′ are the double points of this involution. Hence AB and C′D′ cut II′ dividing it harmonically. Therefore N (ZZ′, II′) = — 1 ; whence N is also a point on the conic. By similar reasoning it can be shown that L, M, L′, M′, N′ are also on R ... (1 4)

Hence Theorem (A) is completely proved.

Cor. The tretrad (ABCD) is also homographic with each of the tetrads (B′A′D′C′), (C′D′A′B′) or (D′C′B′A′). Hence by application of theorem A, the same conic R passes through the poles P_1, P_2, P_3 of these three homographic axes also ... (1.5)

Cor. If the poles of AA′, BB′, CC′, DD′ for S be denoted by $\alpha, \beta, \gamma, \delta$ it follows that the conic R is the pole-locus of the corresponding homographic axis for all conics passing through $\alpha\beta\gamma\delta$. Similarly it is seen that R is the pole locus of the corresponding homographic axis for all conics passing through the poles of joins of corresponding points of ABCD and any of the tetrads homographic with it mentioned in the above corollary.*

Cor. When the two tetrads are in involution, the conic R degenerates into a line pair one of which is the axis of involution. Hence the six meets like (AB, C′D) and the centre of the involution are collinear. It may be noticed that P_1, P_2, P_3 mentioned in (1.5) coalesce with P.

2. It remains to notice certain special cases. To begin with, we may note that as the two triads ABC, A′B′C′ determine uniquely the homographic ranges, the theorem (A) can be restated in the useful form

If two triangles ABC and A′B′C′ are inscribed in a conic S and D′ be any point on the conic, then the meets of D′A′, D′B′ and D′C′ with BC, CA, AB respectively and the meets of the corresponding sides of the two triangles lie on a conic passing through the pole of the Pascal line of the hexagon AB′CA′BC′ ... (2.1)

Further, if any straight line meets the sides BC, CA, AB of a triangle inscribed in a conic S, in L, M, N and if D′ be any point on the conic, then D′L, D′M, D′N meet the conic again in A′B′C′ such that the triangles ABC and A′B′C′ are in perspective. Moreover the centre of perspective lies on the straight line. The converse of this theorem is also true ... (2.2)

The theorem (2.2) gives other particular cases. When S is a circle, the pedal line and Poncelet's theorems are seen to be deductions.

As another example we may consider this theorem given in Lachlan's *Pure Geometry* :—O is a point on the circle circumscribing a triangle ABC. Points A′, B′, C′ are taken on the sides such that AA′, BB′, CC′ subtend right angles at O. Show that A′, B′, C′ are collinear and the line of collinearity passes through the centre of the circle. The truth is at once seen from the converse of the theorem (2.2).

* This gives another method of proof of theorem (A).

3. I have considered some well-known theorems which are deducible from (2.2). The following appear to be both new and interesting:—

If A, B, C, P are four concyclic points and lines PL, PM, PN are drawn making the same angles θ in the same sense with the sides of the triangle ABC, meeting the sides in L, M, N and any conic through ABCP in A′B′C′, then the triangles ABC and A′B′C′ will be always in perspective. Also the centre of perspective traces the line LMN as the conic is varied. For, since L, M, N are collinear, therefore the two triangles ABC and A′B′C′ are in perspective from (2.2). The centre of perspective always lies on this line LMN.

Question 1662 of this journal is another deduction from (2.2).

4 Lastly there are two particular cases of theorem (A) which are worthy of notice.*

When the points I and I′ where R cuts the homographic axis are projected to circular points, the interesting theorem given in Q. 1433 is obtained. The circle R in this case becomes the nine-point circle of the ortho-centric tetrad $\alpha\beta\gamma\delta$ mentioned in (1.6) (4.1)

Secondly when H, H′ are projected into circular points, we get a theorem concerning two directly similar quadrangles inscribed in a circle, the R conic in this case becoming a rectangular hyperbola ...(4.2)

---

* Mr V. Ramaswamy Aiyar drew my attention to Q. 1433 of which theorem (A) is the generalisation and also to the deduction (4.2).

ON A CERTAIN PROBLEM IN DETERMINING THE EXTREMA OF A FUNCTION OF SEVERAL VARIABLES.

By A. Narasinga Rao,

Annamalainagar.

1. *The Problem Stated : Existence Theorem.*

In Goursat's classical treatise on Mathematical Analysis,* one finds an analytical discussion of the problem of determining the position of a point P the sum of whose distances from the vertices of a triangle is a minimum. The object of this note is to state the problem in mechanics equivalent to the generalised problem and by this means discriminate between the proper solutions when P is distinct from any of the vertices, and the improper solutions when it coincides with one of them.

The generalisation we shall consider may be stated thus :

Given any number of points $A_1, A_2 \ldots A_n$, not necessarily co-planar and a single valued and continuous function $f(r)$, it is required to determine the position of a point P such that

$$\phi(P) = a_1 f(r_1) + a_2 f(r_2) \ldots + a_n f(r_n)$$

shall be a minimum, $a_1, a_2 \ldots$ being positive constants and $r_1, r_2 \ldots$ the distances $PA_1, PA_2, \ldots$ etc. ... (1.1)

We shall confine ourselves to the case when $f(r)$ increases monotonically and becomes infinite with r, and $f(r) > 0$ for $r > 0$. The totality of values $\phi(P)$ is bounded below since $\phi(P) \geqslant 0$, and has hence a lower limit. This limit must be actually attained at one or more points P_0 since $\phi(P)$ is a continuous function of position. As the number of vertices A_r is finite, there exists a convex polygon (if the points are co-planar) or a convex polyhedron whose vertices are all included in A_r and such that every A_r is either an interior point or is situated on a face or edge or at a vertex. Calling this polyhedron (or, polygon) Δ, it is clear that P_0, a position of P for which $\phi(P)$ is a minimum,

* Goursat : *Mathematical Analysis*, Translated by Hedrick, Vol I, Ch III, Art. 62.

cannot be outside Δ for then all the points A_r lie on the same side of a plane π through P_0. An infinitesimal movement of P towards this side would diminish all the distances PA_r and hence also the value ϕ(P). The same argument shows also that P_0 cannot be on a face or edge of Δ, for an infinitesimal movement of δs towards the interior would diminish all the distances PA_r by an amount of the same order as δs, except in the case of the A-points lying in the face or edge whose distances are increased by an infinitesimal of order $(\delta s)^2$.

We thus have two types of solutions according as P_0 is an interior point of Δ other than the A-points (proper solution) or is coincident with a vertex of Δ (improper solution). ... (1.2)

2. *The Equivalent Problem in Mechanics.**

Consider n gravitational masses a_1, a_2 ... situated at A_1, A_2, ... and let the law of attraction between two masses at distance r be $mm' f'(r)$. We have then a field of force in which the potential energy of a unit mass at P is ϕ(P) + const. Hence a minimum value of ϕ(P) corresponds to a position of stable equilibrium at P. Our existence theorem proves that there is at least one position of stable equilibrium.

The simplest case is the one discussed by Goursat in which $f(r) = r$, and the force along PA_r is accordingly a_r. We thus have the following experimental method of determining P_0: Take a smooth table with small holes at $A_1, A_2, \ldots A_n$ through each of which pass strings carrying hanging weights $a_1, a_2, \ldots a_n$. Let the other ends of the strings (those on the table) be knotted together at P. The positions of stable equilibrium give the points for which $\Sigma\, a_k r_k$ is a relative minimum (*i.e.*, less than at all other points in its immediate neighbourhood).†

If $n = 3$ and $a_1 = a_2 = a_3$ (so that the weights are equal) we have Fermat's problem discussed by Goursat. It is clear from statical

* The advantages of the introduction of the energy concept in the discussion of this problem was pointed out to me by Mr. V. Ramaswami Aiyar.

† If l be the length of each of the strings, the depth of the centre of gravity of the system below the table is $\sum_k a_k (l - r_k)$ which is a maximum when $\Sigma\, a_k r_k$ is a minimum.

considerations that the strings will be equally included so that P_0 (the Fermat Point*) is one at which each of the sides subtends an angle of 120°.†

If one of the angles of the triangle, say A_1, exceeds 120°, there is then no such point and we have an improper solution with P_0 at A_1.

More generally, if $n = 3$ and a_1, a_2, a_3 are unequal, P_0 is a point (if such exists) such that P_0A_1, P_0A_2, P_0A_3 are parallel to the sides of a triangle (of forces) whose lengths are a_1, a_2, a_3; so that the angles subtended at P_0 are the supplements of the angles of the triangle $a_1a_2a_3$.‡

For a convex quadrilateral $n = 4$ and let $a_1 = a_2 = a_3 = a_4$. We have a rhombus for the force polygon so that the strings are collinear in pairs. Hence P_0 is the intersection of the diagonals and there is no other proper solution.

In the case of a triangle, we saw that improper solutions arise owing to there being no point P_0 at which the sides subtend the proper angles. Their occurrence may also be explained statically as due to one of the masses, say a_r, being too large to allow of equilibrium except when P is at A_r. Thus suppose a_r is greater than the magnitude of the vector sum of forces a_1 along $A_r A_1$, a_2 along $A_r A_2$, ... a_{r-1} along $A_r A_{r-1}$, a_{r+1} along $A_r A_{r+1}$ ... and a_n along $A_r A_n$.

It is then clear that if P is displaced from A_r, the weight a_r will drag the knot P back to A_r. Thus with three points there will be an improper solution with P_0 at A_1 when

$$a_1^2 > a_2^2 + a_3^2 - 2a_2 a_3 \cos A_2\hat{A}_1A_3,$$

that is, when $\angle A_2A_1A_3$ is less than the angle opposite a_1 in a triangle with sides a_1, a_2, a_3. If $a_1 = a_2 = a_3$ we get Goursat's criterion for an improper Fermat Point, namely, one of the angles of $A_1A_2A_3$ exceeding 120°.

---

* *Vide* Edwards: *Differential Calculus*, 1929, p. 430, Ex. 18.

† For four points A, B, C, D, not in a plane, the Fermat point P_0 is such that any pair of opposite edges subtend equal angles at P_0 (Wolstenholme: Problem No. 2229).

‡ *Vide* S. Subramanyan : *Annamalai University Journal*, Vol. III, p. 60.

For n points, the required criterion for an improper solution at A_1 is that a_1 is greater than the resultant of forces a_2, a_3 ... ... along A_1A_2, A_1A_3, ... *

The plane problem with n points A_r is also that of minimising the total distance of travel of populations $a_1, a_2, \ldots$ at A_1 A_2, ... to a common place P_0, under the somewhat artificial assumption that each person travels straight from his home to P_0, and has attracted some attention for $n = 3$ in the *Annamalai University Journal.*† The statical criteria obtained in this paragraph apply equally to the geometrical problem with the A-points distributed in space.

3. When $f(r) = r^2$, matters are considerably simpler. The force system consists of forces $a_k r_k$ along PA_k whose resultant is a force $(a_1 + \ldots a_n)$ PG where G is the centroid of masses $a_1, a_2 \ldots$ at $A_1, A_2 \ldots$ Hence

If $f(r) = r^2$, there is only one position of P which minimises ϕ (P), namely, the centre of mean position of A_1, A_2, ... for multiples $a_1, a_2, \ldots$

Other forms of $f(r)$ may be discussed similarly, but the results are not easily expressible in terms of well-known geometrical entities.

4. Lastly, it remains to point out that while proper solutions correspond to positions P_0 for which $\phi(P) - \phi(P_0)$ is an infinitesimal of a higher order than the distance PP_0 and hence satisfy the relations

$$\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 0$$

where $\phi(P) = \phi(x, y, z)$; this is not true of improper solutions. Thus in a problem with an improper solution at A_1, let P be a point very near A_1 on the locus

$$a_2 r_2 + a_3 r_3 + \ldots a_n r_n = a_2 A_1 A_2 + a_3 A_2 A_3 \ldots + a_n A_1A_n.$$

* An obvious case when this happens is when one of the weights is greater than the sum of all the others ; *Vide* S. Subramanyan : *Annamalai University Journal*, Vol. III, p. 61.

† S. Subramanyan, *loc cit.*

G. V. Krishnaswami : A note on the Centre of Population, *Annamalai University Journal*, Vol. III, pp. 278-282.

G. V. Krishnaswami : *Annamalai University Journal*, Vol. IV, No. 1.

which obviously passes through A_1. Then $\delta\phi(P) = \phi(P) - \phi(A_1) = a_1 r_1$ approximately and this is of the same order as PA_1. Hence the first partial derivatives do not vanish at A_1.

For the plane problem, $\phi(P) = \phi(x, y)$ is a function of position given by

$$\phi(x, y) = \sum_{1}^{n} a_r \sqrt{(x-\xi_r)^2 + (y-\eta_r)^2} \qquad (r = 1, 2, \ldots n)$$

where ξ_r, η_r are the co-ordinates of A_r and x, y those of P. The surface $z = \phi(x, y)$ is found to have conical points corresponding to the points $A_r(\xi_r, \eta_r)$. The point on the surface nearest to the xy plane may be either an ordinary point at which the tangent is parallel to the xy plane (proper solutions) for which

$$\frac{\partial\phi}{\partial x} = 0, \qquad \frac{\partial\phi}{\partial y} = 0,$$

or one of the conical points, in which case the partial derivates do not vanish.

ON CERTAIN CONGRUENCES.

By Pandit Hemraj,
Dyal Singh College, Lahore.

1. Let $\lambda(p^\alpha) = \phi(p^\alpha)$, p being an odd prime;

$\lambda(2^\alpha) = \phi(2^\alpha)$, $\alpha \leqslant 2$; $\lambda(2^\alpha) = \frac{1}{2}\phi(2^\alpha)$, $\alpha > 2$;

and $\lambda(m) =$ the L. C. M. of $\lambda(p_1^{\alpha_1})$, $\lambda(p_2^{\alpha_2})$, ... $\lambda(p_\mu^{\alpha_\mu})$ where $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots p_\mu^{\alpha_\mu}$ (p's being different primes).

We know that if a be prime to m, $a^{\lambda(m)} \equiv 1 \pmod m$.

If d be any divisor of $\phi(p^\alpha)$ where p is a prime, the number of ntegers less than and prime to p^α belonging to exponent d mod p^α is $\phi(d)$.

If l be the L. C. M. of $d_1, d_2, d_3, \ldots d_\mu$, where d_1 is a divisor of $\phi(p_1^{\alpha_1})$, d_2 of $\phi(p_2^{\alpha_2})$, ... d_μ of $\phi(p_\mu^{\alpha_\mu})$, the number of integers

less than and prime to m belonging to exponent d mod m is

$$\Sigma\ [\phi\ (d_1).\phi\ (d_2)\ \ldots\ \phi(d_\mu)].$$

For, if $a_1, a_2 \ldots a_{\phi(d_1)}$ be the integers less than and prime to $p_1^{\alpha_1}$ belonging to exponent d_1 mod $p_1^{\alpha_1}$, they satisfy the congruence $x^{d_1} \equiv 1$ mod $p_1^{\alpha_1}$. And if $b_1, b_2 \ldots b_{\phi(d_2)}$ be such integers for $p_2^{\alpha_2}$, they satisfy the congruence $x^{d_2} \equiv 1$ mod $p_2^{\alpha_2}$.

Now, if y and z be so chosen that $a_r + p_1^{\alpha_1} y = b_\rho + p_2^{\alpha_2} z$ where a_r, b_ρ are numbers of the above sets, the number in either member of this equation will be a common root of the above congruences. Hence if l' be the L. C. M. of d_1 and d_2, the congruence $x^{l'} \equiv 1$ mod $p_1^{\alpha_1} p_2^{\alpha_2}$ has $\phi\ (d_1).\phi\ (d_2)$ roots less than and prime to $p_1^{\alpha_1} p_2^{\alpha_2}$ belonging to exponent l' mod $p_1^{\alpha_1} p_2^{\alpha_2}$-roots arising from the divisors d_1 and d_2. Similar remarks hold for the divisors d'_1 and d'_2 of $\phi\ (p_1^{\alpha_1})$ and $\phi(p_2^{\alpha_2})$ having l' as their L. C. M. Thus the number of integers prime to and less than $p_1^{\alpha_1} p_2^{\alpha_2}$ belonging to exponent l' mod $p_1^{\alpha_1} p_2^{\alpha_2}$ is $\Sigma\ \phi\ (d_1).\phi\ (d_2)$. This can be easily extended.

It follows that if $l = \lambda(m)$, the number of the primitive λ-roots of m is $N = \Sigma\ [\phi\ (d_1).\phi\ (d_2)\ \ldots\ \phi\ (d_\mu)]$ ... (A)

and the number of the primitive λ-roots of m corresponding to the L. C. M. $\lambda\ (m)$ of $\lambda\ (p_1^{\alpha_1}),\ \lambda\ (p_2^{\alpha_2}),\ \ldots\ \lambda\ (p^{\alpha_\mu})$ is

$$N' = \phi\ [\phi\ (p_1^{\alpha_1})] . \phi\ [\phi\ (p_2^{\alpha_2})]\ \ldots\ \phi\ [\phi\ (p_\mu^{\alpha_\mu})] \ldots \text{(B)}$$

The primitive λ-roots of m in (B) may be called proper primitive λ-roots and others improper.

The primitive λ-roots of m can be divided into $\{N/\phi\ [\lambda(m)]\}$ sets, and those of any set can be derived from any root of that set. For example, let $m = 65$. $\lambda\ (m) = 12$. The divisors of $\phi\ (5)$ are $d_1 = 1,\ d_2 = 2,\ d_3 = 4$; and divisors of $\phi\ (13)$ are $\delta_1 = 1,\ \delta_2 = 2,\ \delta_3 = 3,\ \delta_4 = 4,\ \delta_5 = 6,\ \delta_6 = 12$, The L. C. M. of d_3 and δ_6; d_3 and δ_5; d_3 and δ_3; d_2 and δ_6; d_1 and δ_6 is $12 = \lambda\ (m)$. The number of the primitive λ-roots of 65 is

$$\phi\ (d_3) . \phi\ (\delta_6) + \phi\ (d_3) . \phi\ (\delta_3) + \phi\ (d_2) . \phi\ (\delta_6) + \phi\ (d_1) . \phi\delta_6)$$
$$= 2.4 + 2.2 + 2.2 + 1.4 + 1.4 = 24.$$

They are	(1)	2, 32, 63, 33	(corresponding to d_3 and δ_6)
	(2)	7, 37, 58, 28	(„ „)
	(3)	17, 62, 43, 23	(„ d_3 and δ_5)
	(4)	42, 22, 3, 48	(„ d_3 and δ_3)
	(5)	54, 19, 59, 24	(„ d_2 and δ_6)
	(6)	41, 6. 46, 11	(„ d_1 and δ_6)

They are easily obtained by solving certain indeterminate equations of the first degree. Thus

2 belongs to d_3 $(=4)$ mod 5, hence 2 is a root of $x^4 \equiv 1 \bmod 5$.

4 belongs to δ_5 $(=6)$ mod 13, hence 4 is a root of $x^6 \equiv 1 \bmod 13$.

Choose y and z so that $2 + 5y = 4 + 13z$. The number in either member of this equation is a common root of the above congruences. Hence 17 is a root of $x^{12} \equiv 1 \bmod 65$, *i.e.*, it is a primitive λ-root of 65.

Similarly the primitive roots of $2p^{\alpha}$ where p is an odd prime can be deduced from those of 2 and p^{α}.

2. Since $a^{\lambda(m)} \equiv 1 \bmod m$ where a is prime to and less than m, the congruence $ax \equiv -1 \bmod m$ has a unique solution $x = -a^{\lambda(m)-1}$ $= -\mu m + a'$, that is, $x = a'$ where a' is less than and prime to m. Hence integers prime to and less than m can be combined in pairs, such that

$$aa' \equiv -1 \bmod m \qquad \ldots \quad \text{(C)}$$

unless $a' = a$, that is, unless $x^2 + 1 \equiv 0 \bmod m$. Now every odd prime divisor of $x^2 + 1$ where x is a positive integer is of the form $4n + 1$; and $x^2 + 1 \not\equiv 0 \bmod 4$. It follows that the congruence $x^2 + 1 \equiv 0 \bmod m$ has no solution when m is a prime of the form $4n - 1$ or has a prime factor of this form or $m = 2^{\mu}$ $(\mu > 1)$ or has a factor of this form. Thus if P denote the product of numbers less than and prime to m, then having due regard to the number of pairs of these numbers, we have the following :—

(i) $P \equiv -1 \bmod p$	(ii) $P \equiv -1 \bmod 4$
(iii) $P \equiv -1 \bmod p^{\alpha}$	(iv) $P \equiv -1 \bmod 2p^{\alpha}$
(v) $P \equiv -1 \bmod 2^{\mu}$ $(\mu > 2)$	(vi) $P \equiv 1 \bmod m$

where m has a factor of the form p^{α} or 2^{μ} $(\mu > 1)$ p being a prime of the form $4n - 1$ in all cases.

3. Since the congruence $x^2 + 1 \equiv 0 \mod m$ cannot have more than two roots when m is a prime, it follows that if it has a root a, then it must have two roots a and $m - a$. It is easy to show that when m is a prime of the form $4n+1$, the congruence $x^2+1 \equiv 0 \mod m$ has always two roots, one even and the other odd. If $4n$ is of the form r^2 then r and $m - r$ are the solutions.

If g be a primitive root of a prime $4n + 1$, then

$$g^{2n} - 1 \not\equiv 0 \mod (4n + 1).$$

But since $g^{4n} - 1 \equiv 0 \mod (4n + 1)$, it follows that

$$g^{2n} + 1 \equiv 0 \mod (4n + 1) \qquad \ldots \quad \text{(D)}$$

Hence one root of the congruence $x^2 + 1 \equiv 0 \mod (4n + 1)$ is g^n which is congruent to a mod $(4n + 1)$ where $a < 4n + 1$. The other root is $(4n + 1) - a$.

Now 1 and $4n - 1$ are the least and greatest numbers less than and prime to $4n$. The numbers prime to and less than $4n$ can therefore be equally divided into two sets, one containing numbers of the form $1 + 4\lambda$, and the other containing numbers of the form $4n - 1 - 4\mu$.

Thus the primitive roots of $4n + 1$ falling in the first and second sets are of the form $g^{4\lambda+1}$ and $g^{4n-1-4\mu}$ where g is any primitive root of $4n + 1$. Since

$$(g^n)^{4\lambda+1} - (g^n)^{4\lambda'+1} = g^{(4\lambda'+1)n}\,(g^{4(\lambda-\lambda')n} - 1) \equiv 0 \mod (4n+1),$$

and $$(g^n)^{4n-1-4\mu} - (g^n)^{4n-1-4\mu'} \equiv 0 \mod (4n + 1),$$

the two roots are unique.

If γ and γ' be the roots of $x^2 + 1 \equiv 0 \mod (4n + 1)$, then

$$\gamma\gamma' \equiv 1 \mod (4n + 1). \qquad \ldots \quad \text{(E)}$$

It follows from (C) that

(vii) $$4n\,! \equiv -1 \mod (4n + 1).$$

If x be an even root, $x^2 + 1 = my$, $m > y + 1$.

$$\therefore \quad (m - x)^2 + 1 = 2mz \qquad m > z \geqslant 1$$

where m is a prime $4n+1$ and y, z are primes or products of primes of the same form.

Hence $$2x - y = m - 2z \quad \ldots \quad (F)$$

In simple cases, a few trials give x and hence g.

4. If g be a primitive root of p^α where p is a prime of the form $4n+1$, then $g^{\phi(p^\alpha)} - 1 \equiv 0 \bmod p^\alpha$.

$\therefore \quad g^{\frac{1}{2}\phi(p^\alpha)} - 1 \not\equiv 0 \bmod p^\alpha$ and hence $g^{\frac{1}{2}\phi(p^\alpha)} + 1 \equiv 0 \bmod p^\alpha \ldots$ (G)

It follows that $g^{\frac{1}{4}\phi(p^\alpha)}$, that is, $g^{np^{\alpha-1}}$ is a root of the congruence $x^2+1\equiv 0 \bmod p^\alpha$. Hence $p^\alpha - a$ is the other root where $a \equiv g^{np^{\alpha-1}} \bmod p^\alpha$ and $a < p^\alpha$. The numbers less than and prime to $4np^{\alpha-1}$ are of the form $1+4\lambda$ and $4np^{\alpha-1}-1-4\mu$; and therefore the primitive roots of p^α can be divided into two sets as before.

Thus $x^2+1\equiv 0 \bmod p^\alpha$ has two roots unique. If γ and γ' be the roots $$\gamma\gamma' \equiv 1 \bmod p^\alpha. \quad \ldots \quad (H)$$

Similar remarks hold for $x^2+1\equiv 0 \bmod 2p^\alpha$, where p is a prime of the form $4n+1$.

It follows from (C) and (H) that if P be the product of the numbers prime to and less than p^α (or $2p^\alpha$), then

(viii) $$P \equiv -1 \bmod p^\alpha \text{ (or mod } 2p^\alpha).$$

5. Consider the congruence $x^2+1\equiv 0 \bmod m$ where $m = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_\mu^{\alpha_\mu}$ (the p's being different odd primes).

If $\alpha \geqslant 2$ or if any odd prime be of the form $4n-1$, it has no root. It follows from (C) that if P be the product of numbers less than and prime to m in this case, then

(ix) $$P \equiv 1 \bmod m.$$

If $\alpha = 0$ or 1 and each p be of the form $4n+1$, and g be a proper primitive λ-root of m, then $g^{\frac{1}{2}\lambda(m)} - 1 \not\equiv 0 \bmod m$.

But $g^{\lambda(m)} - 1 \equiv 0 \bmod m$. $\therefore\ g^{\frac{1}{2}\lambda(m)} + 1 \equiv 0 \bmod m \ldots$ (I)

$\therefore\ g^{\frac{1}{4}\lambda(m)}$ is a root of $x^2+1\equiv 0 \bmod m$. $m-a$ is another root where $a \equiv g^{\frac{1}{4}\lambda(m)} \bmod m$, a being prime to and less than m.

If γ, γ' be the roots, $\gamma\gamma' \equiv 1 \bmod m$ ... (J)

The number of proper primitive λ-roots of m which satisfy the congruence

$$x^{\frac{1}{2}\lambda(m)} + 1 \equiv 0 \bmod m \text{ is } \mathrm{N}' = \phi\,[\phi\,(p_1^{\alpha_1})] \dots \phi\,[\phi\, p_\mu^{\alpha_\mu})].$$

It may be pointed out that the improper primitive λ-roots of m do not satisfy this congruence, for they are derived from the congruences of degrees lower (at least one being of lower degree) than

$$\phi\,(p_1^{\alpha_1}),\ \phi\,(p_2^{\alpha_2}), \dots \phi\,(p_\mu^{\alpha_\mu}).$$

The proper primitive λ-roots fall in pairs such that the numbers in each pair raised to the power $\frac{1}{4}\lambda(m)$ leave the same remainder when divided by m.

Hence $\frac{1}{2}\mathrm{N}'$ is the number of roots of $x^2 + 1 \equiv 0 \bmod m$, and these roots also fall in pairs such that the product of the numbers in each pair is congruent to 1 mod m.

Since $\mathrm{N}'/4$ (the number of such pairs) is always even, it follows from (C) that if P be the product of numbers prime to and less than m,

(x) $$\mathrm{P} \equiv 1 \bmod m.$$

NOTE. γ, γ' are, in general, other than the primitive roots of p^α (or $2p^\alpha$) or other than the primitive λ-roots of m such that

$$\gamma\gamma' \equiv 1 \bmod p^\alpha \text{ (or } 2p^\alpha) \text{ or mod } m.$$

It may be noted that the primitive roots or primitive λ-roots fall in pairs in two ways such that $gg' \equiv -1$ and $gg_1 \equiv 1$.

6. Let $a_1, a_2, a_3 \dots a_n$ be the numbers less than and prime to p^α (or $2p^\alpha$). Then it can be proved that

$$\left.\begin{array}{l} x^{\phi(p^\alpha)} - 1 \equiv (x - a_1)(x - a_2) \dots (x - a_n) \bmod p^\alpha \\ x^{\phi(2p^\alpha)} - 1 \equiv (x - a_1)(x - a_2) \dots (x - a_n) \bmod 2p^\alpha \end{array}\right\} \dots \text{(K)}$$

where p is an odd prime.

7. The theorem due to Gauss which was partially proved in my note 'On an extension of Wilson's Theorem' in *J. I. M. S.*, Vol. XIX, No. 2, has been proved above, but the following proof appears to be interesting.

(*a*) Let $m = p^{\mu}$ where p is a prime not less than 3 and $\mu > 1$, and $\phi(p^{\mu}) = n$. The integers prime to and less than m are

$$rp + 1,\ rp + 2,\ rp + 3, \ldots rp + p - 1 \qquad \ldots \quad (1)$$

where r varies from 0 to $p^{\mu-1} - 1$.

The numbers in (1) are a complete set of roots of the congruence $x^n - 1 \equiv 0 \bmod m$. If γ be a primitive root of m, then the numbers

$$\gamma,\ \gamma^2,\ \gamma^3, \ldots \gamma^n \qquad \ldots \quad (2)$$

are also all the incongruent roots of the congruence.

The numbers in (1) are therefore congruent in some order to the numbers in 2).

Denoting the numbers in (1) by $\alpha_1, \alpha_2, \ldots \alpha_n$, we have (say)

$$\alpha_1 \equiv \gamma \bmod m;\ \alpha_2 \equiv \gamma^2 \bmod m;\ \ldots\ \ldots\ \alpha_n \equiv \gamma^n \bmod m.$$

$$\therefore\ \alpha_1 \alpha_2 \ldots \alpha_n \equiv P \equiv \gamma^{\frac{n}{2}(n+1)} \equiv (\gamma^n)^{\frac{n}{2}} . (\gamma^n)^{\frac{1}{2}} \bmod m$$

where P is the product of numbers in (1); Hence $P - (1)^{\frac{1}{2}} \equiv 0 \bmod m$,

i.e., $\quad P + 1 \equiv 0$ or $P - 1 \equiv 0 \bmod m$.

Now the only term in P not divisible by p is $\{(p-1)!\}^{m/p}$ and in $P - 1$, the term

$$\{(p-1)!\}^{m/p} - 1 = [(p-1)! - 1]\,[(p-1!)^{m/p-1} + \ldots +$$

$$\ldots + (p-1)! + 1] = [(p-1)! - 1]\,F(p).$$

$(p-1)! - 1 \not\equiv 0 \bmod p$, and the number of terms in $F(p)$ is odd and when they are combined in consecutive pairs from the end or

beginning, we see that F (p) is also not divisible by p. In fa t

$$\{(p-1)!\}^{p^{\mu-1}} = hp^{\mu} - 1.$$

$\therefore$ $P - 1 \not\equiv 0 \bmod m$.

Hence it follows that $P + 1 \equiv 0 \bmod p^{\mu}$. Thus the product of numbers prime to and less than p^{μ} is congruent to $-1 \bmod p^{\mu}$.

(b) Let $m = 2p^{\mu}$ where p is a prime $\geqslant 3$ and $\mu \geqslant 1$, and $\phi(2p^{\mu}) = n$. The integers prime to and less than m are

$$\left.\begin{array}{l} rp + 1,\ rp + 3,\ \ldots\ rp + p - 2 \\ tp + 2,\ tp + 4,\ \ldots\ tp + p - 1 \end{array}\right\} \quad \ldots \quad (3)$$

where r is 0 or runs through even values from 2 to $2(p^{\mu-1} - 1)$ and t runs through odd values from 1 to $2p^{\mu-1} - 1$. The numbers in (3) are a complete set of roots of $x^n - 1 \equiv 0 \bmod m$. If γ be a primitive root of m, then $\gamma, \gamma^2, \ldots \gamma^n$ are also all the incongruent roots of the congruence. Denoting the numbers in (3) by $a_1, a_2, \ldots a_n$ and their product by P, we have as before $P - 1 \equiv 0$ or $P + 1 \equiv 0 \bmod m$. Either is congruent to 0 mod 2.

The only term in P not divisible by p is $(p-1)!^{p^{\mu-1}}$.

As before, $\qquad P + 1 \equiv 0 \bmod 2p^{\mu}$.

(c) Let $m = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{\mu}^{\alpha_{\mu}}$ where p's are odd primes, and $\phi(m) = n$. $n \equiv 0 \bmod 4$. The integers prime to and less than m are

$$\beta_1, \beta_2, \ldots \beta_n \quad \ldots \quad (4)$$

The primitive λ-roots of m can be divided into $N/\phi\,[\lambda(m)]$ sets, and the product of the primitive λ-roots of any set and hence the product of all the primitive λ-roots of m is congruent to 1 mod m, provided that $\lambda(m) > 2$. If N be the number of the primitive λ-roots of m, $N \equiv 0 \bmod 4$. The primitive λ-roots of m are some numbers of (4). The remaining numbers of (4) fall in pairs such that the product of the numbers in each pair is congruent to — mod m. See (C).

For if a be any one of the remaining numbers, the congruence $ax \equiv -1 \bmod m$ gives $x = -a^{\lambda(m)-1} = -\mu m + a'$, *i.e.*, $x = a' < m$, a' is also one of the remaining numbers. If not, it must be a primitive λ-root of m, *i.e.*, $a', a'^2, a'^3, \ldots a'^{\lambda(m)}$ will, when divided by m, leave different remainders prime to and less than m, *i.e.*, $\{\mu m - a^{\lambda(m)-1}\}^r$ $r = 1, 2, 3, \ldots \lambda(m)$ will, when divided by m, leave different remainders. Hence

$$(-)^r a^{r\lambda(m)-r} = (-)^r \{(a^{(r-1)\lambda(m)} - 1) a^{\lambda(m)-r} + a^{\lambda(m)-r}\},$$

i.e., $(-)^r a^{\lambda(m)-r}$ will leave different remainders.

Hence a must also be a primitive λ-root which is not so. It follows that the product of all the remaining numbers is congruent to 1 mod m. Thus if P is the product of numbers prime to and less than m P $\equiv$ 1 mod m.

NOTE. Remembering that there exist two numbers γ, γ' [other than primitive roots, in general, of p^α (or $2p^\alpha$] prime to and less than p^α (or $2p^\alpha$) when p is a prime of the form $4n+1$, such that

$$\gamma\gamma' \equiv 1 \bmod p^\alpha \text{ or } 2p^\alpha,$$

the cases (a) and (b) also can be similarly proved.

GLEANINGS.

33. Before Fermat, Vieta occupied important positions in the state administration. In more recent times, still in France, we find Bailly and Condorcet, two victims of the French revolution and a little later, Monge and Carnot who played important roles in the same period. If Laplace showed for public affairs absolutely negative qualities, Fourier occupied with general satisfaction, and for a long time, the important post of Prefect. In the biography of Jacobi one has recently discovered a parenthesis of a political character in 1848. Finally if Ch. de Freycinet did not make a considerable mark in our science, it is because he had better developed his talent for public life. One learns of this from the splendid discourse of welcome of M Picard, who succeeded him at the Academie Francaise in 1928.

I shall conclude by remarking that in past decades, the miserable political conditions of my own country did not make it possible for Italian mathematicians to participate in public life; but in more recent times one meets with G. B. Gioigini (friend of Chasles), F. Brioschi, E. Betti, L. Cremona, and U. Dini, who were successful in handling questions relative to the administration of the state.

G LORIA, in *Scripta Mathematica*, Vol. I, March 1933.

A NOTE ON JULIA'S LEMMA.

BY K. VENKATACHALIENGAR, *Institute of Science, Bangalore.*

The following formulation of Julia's Lemma* is easy to prove and appears to be interesting. What is proved here is not the sharpest form of the Lemma as given by Caratheodory†, but this formulation is given at the end of the paper. The proof of this can be easily constructed along the lines of Nevanlinna's proof given in his remarkable paper on bounded functions‡. Many of the results contained there can be proved more easily by formulating the results in the new way.

The new formulation of Julia's Lemma is this

(A) *Let* $w = f(z)$ *be defined and analytic in* $R(z) > 0$, *and let* $R(w) \geqslant 0$. *Let* $z_1, z_2, z_3, \ldots, z_n, \ldots$ *be a sequence of points on the positive part of the real axis such that* $z_n \to \infty$ *monotonically, and let* $R(w_n) \geqslant R(z_n)$.

Then $R(w) \geqslant R(z)$ *for all* z, *the equality sign occurring only when* $w = z + \alpha i$, *where* α *is a real constant.*

Proof. Let z'_n and w'_n be the images of z_n and w_n in the imaginary axis. Then $\phi(t) = (w - w_n)/(w - w'_n)$, considered as a function of $t = (z - z_n)/(z - z'_n)$ is regular and $|\phi(t)| < 1$ for $|t| < 1$. Moreover $\phi(0) = 0$. Hence by Schwarz's Lemma $|\phi(t)| \leqslant |t|$ in $|t| < 1$. Now the circle $|t| < 1$ corresponds to $R(z) > 0$. Hence

$$\text{(B)} \qquad \left| \frac{w - w_n}{w - w'_n} \right| \leqslant \left| \frac{z - z_n}{z - z'_n} \right| \text{ if } R(z) > 0.$$

Now, if (A) is not true, there is a point z' such that $R(w') < R(z')$, where $w' = f(z')$. By using (B) we will obtain an absurdity.

Let $k_n = |(z' - z_n)/(z' - z'_n)|$, and O be the origin. Let P be the nearer intersection of the circle $\left| \frac{z - z_n}{z - z'_n} \right| = k_n$ with the

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* Bieberbach : *Lehrbuch der Funktionen theorie* Bd. II, p. 112.

† *Loc. cit.*

‡ *Fest schrift an Lindlof* (Helsingfors, 1930).

real axis. Then $OP = z_n (1 - k_n) / (1 + k_n)$. From (B) we deduce that for all z which belong to the circle C_1 given by $\left| \frac{z - z_n}{z - z'_n} \right| \leqslant k_n$, the corresponding values of w are contained in the circle C_2 given by

$$\left| \frac{w - w_n}{w - w'_n} \right| \leqslant k_n.$$

Let the line $w_n w'_n$ cut the imaginary axis in O′ and the circle C_2 in two points of which the nearer is P′. Then

$$O'P' = R(w_n).\frac{1 - k_n}{1 + k_n} = \frac{R(w_n)}{z_n} OP \geqslant OP.$$

Now the real part of w in C_2 is at least equal to O′P′. Therefore we obtain that the real part of w' is at least equal to OP. Now from elementary geometry it is clear that as $z_n \to \infty$, $OP/R(z') \to 1$. Hence we deduce that the real part of w' is at least equal to the real part of z'. This contradicts our preceding hypothesis. Hence we obtain that $R(w) \geqslant R(z)$ always. Now suppose the equality sign occurs for some point in the half plane $R(z) > 0$. Now consider the function $w - z$. This is regular in $R(z) > 0$, and we have $R(w - z) \geqslant 0$. Now by the maximum-modulus principle, we obtain that the supposition is only true if $w - z = \alpha i$. This completes the proof. The formulation of the Lemma in the sharpest known form is as follows:

With the above notation, let there be a sequence z_n, such that $z_n \to \infty$ and $w_n \to \infty$. Suppose

$$\lim_{n \to \infty} \left| \frac{w_n}{z_n} \right|^2 \frac{R(z_n)}{R(w_n)} = k, \text{ a finite constant.}$$

Then there exists a number $\lambda \geqslant k$, such that in any region in which $I(z) / R(z)$ is bounded, w/z, $R(w)/R(z)$, and $w'(z)$ tend to the same limit λ as $z \to \infty$. [By supposition $R(z) \to \infty$].

ON THE ORTHO POINT-LINE TRANSFORMATION.

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1. In a brief discussion "On conics having a common self-polar triangle" Mr. K. Satyanarayana* has defined the ortho-line of a point P with respect to a triangle ABC as the line of collinearity, p, of the points L, M, N in which the lines through P perpendicular to PA, PB, PC meet the sides of the triangle ABC. That a line p has two ortho-points of which p is the ortho-line—is also evident; for it may be any one of the points in which the circles on AL, BM, CN as diametres meet. The purpose of this note is to draw attention to an alternate definition of the ortho-line of a point and to discuss synthetically a few interesting properties connected with the ortho-line.

2. Let S be the unique conic having ABC for a self-polar triangle and P for a focus. Since PA, PL are perpendicular lines through the focus, the pole of PA lies on PL; but it lies also on BC and is hence the point L. Similar considerations apply to the lines PB and PC. Hence L, M, N are collinear and the line of collinearity, p, is the directrix corresponding to P of S. Thus:

The ortho-line p of a point P with respect to a triangle ABC is the directrix corresponding to P of the unique conic having ABC for a self-polar triangle and P for a focus ... (2.1)

Consider the pencil of conics through A, B, C, P and let H be the unique rectangular hyperbola in the pencil. By reciprocating with respect to a circle with centre at P and denoting the reciprocal elements by corresponding small letters, we see that the Fregier point of P for a conic Σ through A, B, C, P will reciprocate into the directrix of a parabola σ touching the lines a, b, c. Since these directrices pass through the ortho-centre of the triangle formed by the lines a, b, c that is, the centre of the unique circle s having abc for a self-polar

* *Vide The Math. Student*, Vol I, No. 1 March (1933), pp. 19—24.

triangle, the locus of Fregier points of P for all conics through A, B, C, P is the polar of P with respect to S. Hence

The ortho-line of P with respect to a triangle ABC is the locus of Fregier points of P for all conics through A, B, C, P ... (2.2)

In particular, considering the rectangular hyperbola H, since any chord of H subtending a right angle at P must be perpendicular to the tangent at P to H, and therefore parallel to the normal at P, the Fregier point of P for H is the point at infinity on the normal at P to H. Thus:

The ortho-line of P is parallel to the normal at P to the unique rectangular hyperbola through A, B, C, P ... (2.3)

If D is the ortho-centre of ABC, the rectangular hyperbola H is the same for the four triangles formed out of A, B, C, D. Further, H passes through the ortho-centres O_{22} of PBC, O_{11} of PAD, etc. Since AD is perpendicular to BC, the chord O_{2},O_{14} of H subtends a right angle at P and is, therefore, parallel to the normal at P to H. Thus:

The ortho-lines of P with respect to the four triangles formed out of an ortho-centric tetrad ABCD and the joins of the ortho-centres of pairs of triangles like PBC, PAD, etc., are all parallel ... (2.4)

Let PT be the tangent at P to H, and p_∞ the point at infinity on p. Now $(L, M, N, p_\infty) = P(L, M, N, p_\infty) = P(A, B, C, T) = D(A, B, C, P)$ (by Chasles' theorem) is a constant if P moves on a line through D. Hence:

The ortho-lines of the points of a line through the ortho-centre of the triangle ABC, envelope a parabola inscribed in the triangle ... (2.5)

The points L, M, N are the Fregier points of P for the three line pairs in the pencil of conics through A, B, C, P. Since p is parallel to normal at P to H, the axis of S through P is the tangent at P to H. Further, p meets H in a pair of points which subtend a right angle at P.

3. Now, the ortho-points P, P′ of a line p are the two points of intersection of the circles on AL, BM, CN as diameters. If Γ is the polar circle of ABC, Γ is out-polar to all conics touching p and the sides of ABC and hence cuts all the members of the co-axal system

formed by their director circles orthogonally. Hence Γ belongs to the conjugate co-axal system and, therefore, has the limiting points P, P′ of this system for inverse points. Thus :

The two ortho-points of a line p with respect to a triangle ABC are inverse points with respect to the polar circle of ABC ... (3.1)

If H′ is the unique rectangular hyperbola through A, B, C, P′, the normals at P, P′ to H, H′ are both parallel to p. Hence:

If P and P′ are inverse points with respect to the polar circle of a triangle ABC, the tangents at P, P′ to the rectangular hyperbolas through A, B, C, P ; A, B, C, P′ are parallel ... (3.2)

Let S′ be the unique conic through A, B, C, P, P′. Since S′ is out-polar to Γ, the ϕ-conic of S′ and Γ is the reciprocal of S′ with respect to Γ. But since P, P′ are inverse points with respect to Γ, PP′ must touch the ϕ-conic. Hence, the pole of PP′ with respect to Γ, *viz.*, the point at infinity, P_∞ in the direction perpendicular to PP′ must lie on S′. Also if p meets PP′ in P_0, $(LMNP_0) = P'(LMNP_0) = P'(ABCP_\infty)$ on the conic S′ = the cross ratio of the points in which PP_∞ meets BC, CA, AB, PP′, all these lines being tangents to the ϕ-conic of S′ and Γ. Thus :

If P, P′ are inverse points with respect to the polar circle of a triangle ABC, the ortho-line of P (or P′) touches the ϕ-conics of the polar circle and the unique conic through A, B, C, P, P′ ... (3.3)

Now the chord $P'P_\infty$ of S′ subtends a right angle at P. Hence $P'P_\infty$ meets p in the Fregier point Q of P for S′. Further if PP′ meets BC, CA, AB in A′, B′, C′ respectively,

$$(LMNQ) = P_\infty(ABCP) = (A'B'C'P').$$

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34. God is that actuality in the world, in virtue of which there is physical law.

A. N. WHITEHEAD: *Process and Reality.*

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

Note on Euler's Constant.

The proof given in Chrystal's *Algebra* (p. 81, Part II, first edition) of the existence of this constant leaves something to be desired.

The problem is to show that, as $n \to \infty$, the function

$$S_n \equiv 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n \qquad (n = 1, 2, 3, \ldots) \ldots \quad (1)$$

tends to a limit γ such that $1 > \gamma > 0$.

In the proof given it is shown that S_n, for all values of n however great, lies between 0 and 1 ; and from that it is left to be understood that S_n tends to a limit γ lying between 0 and 1. This is hardly fair to the reader.

As a matter of fact S_n, starting from the value 1, decreases as n increases. The reader is not *helped* to realize this ; but a reader, realizing this, might ask "could not the limit be zero itself ?"

Consider, for example, the function $2^{1/n} - 1$, $(n = 1\ 2, 3, \ldots)$. This also lies between 0 and 1 for values of n however great. But its limit is 0 and not a quantity lying between 0 and 1.

A very informing way of dealing with the point thus raised is to consider the companion function

$$T_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1), \quad (n = 1, 2, 3, \ldots) \ldots \quad (2)$$

Both S_n and T_n can now be treated by means of the inequality used in the proof, which may be written

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n} \qquad \ldots \quad (3)$$

It can be seen from it

(1) that S_n (having the starting value 1) decreases as n increases ;

(2) that T_n has the starting value $1 - \log 2$, which is *positive* ;

(3) that T_n increases as n increases ;

and (4) that $S_n - T_n$ is positive, but tends to 0 as $n \to \infty$.

Hence, it follows that both S_n and T_n tend to a common limit γ such that

$$1 > \gamma > 1 - \log 2 > 0 \quad \ldots \quad (4)$$

It is also seen that *every* S_n is greater than *every* T_n, and that γ lies *in* between them.

V. RAMASWAMI AIYAR.

On the Equation to a pair of tangents to a conic.

1. The following is a simple method for obtaining the combined equation to the tangents from a point P (x_1, y_1) to a conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in the form $SS_1 = T^2$, and compares favourably with the usual text-book methods in point of elegance ... (1)

2. LEMMA. If P (x_1, y_1) and Q (x_2, y_2) are two points through which a pair of parallel lines are drawn cutting (1) in P_1P_2 ; Q_1Q_2 then

$$\frac{PP_1 \cdot PP_2}{QQ_1 \cdot QQ_2} = \frac{S_1}{S_2} \quad \ldots \quad (2)$$

For if θ is the angle made by these parallel lines with the x-axis, PP_1, PP_2, are the roots of the equation

$$r^2 (a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta)$$

$$+ 2r \, [(ax_1 + hy_1 + g) \cos \theta + (hx_1 - by_1 + t) \sin \theta] + S_1 = 0$$

From this and the similar equation for QQ_1, QQ_2 (2) follows.

3. Now let Q be any point on either tangent from P to (1). Let, T be the point of contact of this tangent. Then taking the line PQ, we get from (2)

$$\frac{PT^2}{QT^2} = \frac{S_1}{S_2} \qquad \ldots \quad (3)$$

But the ratio PT/QT is equal to the ratio of the distances of P and Q from the chord of contact of the tangent, from P, *viz.*,

$$T \equiv axx_1 + h\,(xy_1 + x_1y) + byy_1 + g\,(x + x_1) + f\,(y + y_1) + c = 0.$$

Hence $$\frac{PT^2}{QT^2} = \frac{T_1^{\,2}}{T_2^{\,2}} = \frac{S_1^{\,2}}{T_2^{\,2}} \qquad \ldots \quad (4)$$

Equating (3) and (4) and changing (x_2, y_2) into (x, y)

we get $$\frac{S_1}{S} = \frac{S_1^{\,2}}{T^2} \quad \text{or} \quad SS_1 = T^2.$$

Presidency College, Madras. S. Y. MUTHUSWAMY, B. Sc. Class.

On the Multiplication of certain numbers.

Anent Mr. K. Subba Rao's note on the "*Multiplication of numbers ending in 5,*" (*Math. Student,* Vol. II, No. 2, June 1934), the following will be read with interest.

1. The product of two numbers whose last digits total up to 10, but whose other digits are identical, (*i.e.*, numbers of the type ax and, ay, where $x+y=10$, is given by $[(a+1)\times a]p$ where p stands for the actual product $x\times y$.

Thus (i) $194\times196=(20\times19)\,24=98024$,

(ii) $85\times85=(9\times8)\,25=7225.$

2. The special case, where $x=y=5$, [example (ii) above], gives a rule for writing down the square of a number ending in 5, *viz.*, $(a5)\times(a5)=[(a+1)\times a]\,25$. Thus $75^2=5625$ and $155^2=24025$.

3. An extension of the above case, where each number ends in 5, but the other figures are different, is the one given by Mr. Subba Rao;

and the result may be expressed in the following *somewhat* simpler manner.

$$(a5) \times (b5) = \left[a \times b + \frac{a+b}{2} \right] 25,$$

where, should the proper fraction $\frac{1}{2}$ occur while simplifying the terms in the rectangular brackets, it has the effect of increasing the digit in the tens' place by 5, (being in fact equal to half a hundred); *i.e.*, changing 25 to 75.

Thus $$45 \times 95 = \left[4 \times 9 + \frac{4+9}{2} \right] 25 = 4275 \quad \text{and}$$

$$85 \times 155 = \left[8 \times 15 + \frac{8+15}{2} \right] 25 = 13175.$$

These rules are useful when we have to deal with numbers of two or three digits; but they lose much of their utility when applied to numbers having more digits.

Central College, Bangalore. M. V. JAMBUNATHAN, M.A., B.Sc.

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35. **Mathematics and Law.** What men have achieved something of note in each of the fields mathematics and law? Five names occur to one rather readily. The great French mathematician Prine de Pierre (c. 1608—1665) was councillor for the parliament of Toulouse and became noted for his knowledge of law. The versatile and able German mathematician and philosopher Gottfried Wilhelm Freiherr von Leibnitz (1646—1716) spent some time in diplomatic service and published material for a code of international law. The Englishman John Wilson (1741—1793) who was the senior wrangler of 1761, a good teacher of Mathematics, and the discoverer of Wilson's theorem, became a Judge of common pleas (1786—93), and in our own day two more Frenchmen are to be mentioned, Painleve and Borel. Paul Painleve (b. 1863) professor of rational mechanics? ... ... President of the Chamber of Deputies in 1906, minister of public instruction and inventions in the Briand Cabinet (1915-1916), minister of war in the Ribot Cabinet (1917), and also premier for part of the same year in 1924, President of the Chamber of Deputies, in 1925 again Premier, and in 1926—28 minister of war in two successive cabinets formed by Poincare.

Emile Borel (b. 1871). Professor of the Calculus of Probability and of Mathematical Physics at the Sorbonne, was elected to the Chamber of Deputies in 1924 and was minister of marine in Painleve's cabinet in 1925.

R. C. ARCHIBALD, in *Scripta Mathematica*, Vol. I, December 1932.

SOLUTIONS TO QUESTIONS.

Question 1635.

(K. J. SANJANA) :—(i) If n distinct primes of the form $4m + 1$ are multiplied together, show that the product can be resolved into a sum of two squares in 2^{n-1} ways.

(ii) Prove that the nth power of a prime of the form $4m + 1$ can be represented as a sum of two squares in $\frac{1}{2}(n + 1)$ or $\frac{1}{2}n$ ways according as n is an odd or even integer.

Solution by S. Minakshisundaram.

The proofs of the two theorems depend on the fact that any prime of the form $4m + 1$ can be uniquely expressed as the sum of two squares. (See Reid : *Theory of Algebraic Numbers*, p. 171, Art. 9).

(i) Let $p_1, p_2, \dots p_n$ be the n distinct primes of the form $4m + 1$ and $p_r = a^2_r + b^2_r$. It is evident then that the product $p_1 p_2 \dots p_n = \prod_{r=1}^{n} (a_r^2 + b_r^2)$ can be expressed as the product of two conjugate complex numbers in 2^{n-1} ways.

(ii) Again if $p = a^2 + b^2$ then $p^n = (a^2 + b^2)^n$, and

$$p^n = \{(a + bi)^{n-r} (a - bi)^r\} \{(a + bi)^r (a - bi)^{n-r}\}.$$

(a) If n is an even integer, r can be allowed to take all values from 0 to $\frac{1}{2}n - 1$. When $r = \frac{1}{2}n$

$$p^n = \{(a+bi)^{n/2}(a-bi)^{n/2}\} \{(a+bi)^{n/2}(a-bi)^{n/2}\} = (a+bi)^n (a-bi)^n$$

the same as when $r = 0$: and when $r > n/2$ we will obtain one or other of the forms already found. Thus there are $n/2$ ways of expressing p^n as the sum of two squares.

(b) If n is an odd number r can take all values from 0 to $\frac{1}{2}(n-1)$ and when $r > \frac{1}{2}(n - 1)$ we will obtain one or other of the forms already obtained. Hence there are $\frac{1}{2}(n - 1) + 1$ or $\frac{1}{2}(n + 1)$ ways of expressing p^n as the sum of two squares.

A more general result is :—

Let n be a composite number of the form

$$2^{\nu} \prod_{r} (4n_r + 1)^{\alpha_r} \prod_{r} (4n_s - 1)^{\beta_s}$$

where $(4n_r + 1)$ and $(4n_s - 1)$ are primes.

(i) A necessary and sufficient condition that n may be expressed as the sum of two squares is that all the β's are even.

(ii) If the condition is satisfied, the number of ways in which n can be represented as the sum of two squares is

$$\tfrac{1}{2} \prod_{r} (\alpha_r + 1) \text{ or } \tfrac{1}{2} \prod_{r} (\alpha_r + 1) + \tfrac{1}{2}$$

according as some or more of the α's are odd ; (Reid: *Algebraic Numbers*, p. 191 art. 16).

Questions 1641 & 1644.

(C. N. SRINIVASIENGAR) :—A plane curve of degree n has δ nodes, k_1 keratoid cusps, k_2 tacnodes, and k_3 ramphoid cusps. Prove that

(i) $m = n(n-1) - 2\delta - 3k_1 - 4k_2 - 5k_3$.

(ii) $n = m(m-1) - 2\tau - 3i - 4k_2 - 5k_3$.

(iii) $i = 3n(n-2) - 6\delta - 8k_1 - 12k_2 - 15k_3$.

(iv) $i - k_1 = 3(m - n)$.

Solution and remarks by the Proposer.

1. Suppose a plane curve has a cusp of any kind. Taking the origin at the cusp and $y = 0$ as the cuspidal tangent, the cartesian equation of the curve is

$$y^2 = (a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3) + (b_0x^4 + 4b_1x^3y + \dots) + \dots$$

The necessary and sufficient conditions* for a ramphoid cusp at O are (i) $a_0 = 0$. (ii) $y^2 - 3a_1x^2y - b_0x^4 =$ a perfect square, a_1 and b_0 being not zero. (iii) $27a_2a_1^2 - 24a_1b_1 + 4c_0 \neq 0$ Condition (i)

* Use Newton's diagram, *vide* Hilton : *Plane Algebraic Curves.*

without (ii) is the necessary and sufficient condition for a tacnode at O. When $a_0 \neq 0$, the origin is an 'ordinary' or a 'keratoid' cusp.

If O be a tacnode, the expansion of the curve near O consists of two *linear* branches of the form $y = Ax^2 + Bx^3 + \dots$ But if O be a ramphoid cusp, the expansion near O consists of a super-linear (cyclical) branch of order 2, wherein the lowest term in x begins with x^2. It may be written $y = bx^2 \pm cx^{5/2} + dx^3 + \dots$ The expansion for an ordinary keratoid cusp is of the form

$$p = \pm ax^{3/2} + bx^2 \pm cx^{5/2} + dx^3 + \dots$$

It is also known that the reciprocals of a keratoid cusp, a tacnode and a ramphoid cusp are respectively tangents at an inflexion, a tacnode, and a ramphoid cusp (*vide* Hilton *loc. cit.*, Chapter VI, § 5).

2. Plucker's equations, as they are usually written assume that the nodes and cusps are ordinary ones. When a curve possesses a tacnode, a ramphoid cusp, or other higher singularities, Plucker's equations can be retained without being blown up only by a very artificial process, which consists in supposing that any such multiple point O is equivalent to a particular combination of δ ordinary nodes κ ordinary cusps, τ bitangents with coincident points of contact at O, i inflexional tangents with inflexions at O. The numbers δ, κ, τ, i can be determined in each concrete instance by the help of the generalised definition for deficiency, but there is no guarantee that they are all positive integers.

3. Let us start with the general equation

$$y^2z^{n-2} + (a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3)\,z^{n-3}$$
$$+ (b_0x^4 + 4b_1x^3y + 6b_2x^2y^2 - 4b_3xy^3 + b_4y^4)\,z^{n-4} + \dots$$

where $z = 1$. Unless a_0 and a_1 are both zero, the origin is an ordinary point on the first polar of any point and the tangent coincides with the cuspidal tangent. The expansion of the first polar at O is therefore of the form

$$y^2 = A_0x^2 + A_1x^3 + \dots \dots$$

$2A_0$ gives the value of d^2y/dx^2 at (0, 0) for the first polar. The value of d^2y/dx^2 at (0, 0) for the given curve is $-3a_1$ when $a_0 = 0$. It will

be found that $-3a_1 = 2A_0$ if and only if $9a_1^2 = 4b_0$, that is, if O be a ramphoid cusp. Hence, if O be a ramphoid cusp, the coefficients of x^2 in the expansion of the first polar of any point and in that of the super-linear branch of the curve at O are equal. It follows that the number of intersections of the curve and a general first polar curve are four at a tacnode and five at a ramphoid cusp (*vide* Hilton : Chap. VI).

Hence, if a n-ic has δ nodes, k_1 Keraloid cusps, k_2 tacnodes, k_3 ramphoid cusps, we have

$$m = n(n-1) - 2\delta - 3k_1 - 4k_2 - 5k_3 \qquad \dots \quad (1)$$

The reciprocal curve therefore gives

$$n = m(m-1) - 2\tau - 3i - 4k_2 - 5k_3 \qquad \dots \quad (2)$$

Let us assume

$$i = 3n(n-2) - 6\delta - 8k_1 - \lambda k_2 - \mu k_3 \qquad \dots \quad (3)$$

so that

$$k_1 = 3m(m-2) - 6\tau - 8i - \lambda k_2 - \mu k_3 \qquad \dots \quad (4)$$

Consistency requires $\lambda - 24 = -\lambda$; $\mu - 30 = -\mu$.

Eliminate, say, τ and i from (2), (3), (4), and compare with (1).

$\therefore \quad \lambda = 12, \ \mu = 15.$

The last equation $i - k_1 = 3(m - n)$ follows mechanically.

4. The equations (3) and (4) can be worked out directly by consideration of the Hessian. The work is a bit tedious, but it leads to an interesting result.

Unless $a_1 = 0$, a tacnode or a ramphoid cusp of the given curve will be a triple point on the Hessian, at which there are *three coincident tangents*. The co-efficients of x^4, x^3y, and x^5 are all absent in the equation of the Hessian when $a_0 = 0$, $z = 1$. Hence an inspection of Newton's diagram will indicate that the approximation to the curve in the neighbourhood of the origin is

$$y^3 + Ax^2y^2 + Bx^4y + Cx^6 = 0$$

C does not vanish when $a_0 = 0$, unless a_1 and b_0 are also zero. For the tacnode, therefore, the expansion of the Hessian at the origin consists of three linear branches of the form $y = \alpha x^2 + \beta x^3 + \dots$; but

in special cases, there may be one linear branch and a super-linear branch of order 2. The expression $y^3 + Ax^2y^2 + Bx^4y + Cx^6$ becomes a perfect cube if and only if $4b_0 = 9a_1^2$, *i.e.*, if O is a ramphoid cusp. The expression is then

$$\left(y + \frac{3a_1}{2}x^2\right)^3.$$

Hence, for a ramphoid cusp, the expansion of the Hessian is

$$y = A_0x^2 + B_0\omega x^{\frac{7}{3}} + C_0\omega^2 x^{\frac{8}{3}} + \dots \qquad (\omega^3 = 1)$$

where A_0 is also the coefficient of x^2 in the expansion of the given curve at O. It is easily verified that $B_0 = 0$.

It will now be evident that the curve and the Hessian intersect 12 times at a tacnode, and 15 times at a ramphoid cusp.

Hence follow equations (3) and (4). Moreover, since A_0 is the same for the three curves, we have the following theorem :

If a plane curve possesses a ramphoid cusp, the two branches of the curve at the cusp, the first polar of any point not on the cuspidal tangent. and the three partial branches of the Hessian at the cusp, all have the same curvature at the point.

This is the result in Question 1644.

5. A special case of a tacnode is when $a_0 = a_1 = 0$, ($b_0 \neq 0$). The tacnode will be a point of inflexion on the first polar of every point not on the cuspidal tangent, and it will be a quadruple point on the Hessian at which two tangents coincide with the cuspidal tangent. Plucker's equations, as modified above, remain unaffected.

[NOTE. Equation (1) has been deduced by Hilton from the theory of quadrati transformations. (*vide* Examples 3 and 4, p. 134, first edition). But his statement that a ramphoid cusp is equivalent to the combination of a node and an ordinary cusp is misleading, unless it is also accompanied by a certain number of inflexions and bitangents].

Question 1646.

(A. A. KRISHNASWAMY AYYANGAR) :—Show that the equation of the axis of the parabola $S = 0$ in rectangular cartesian co-ordinates can be put in either of the forms $\frac{\partial}{\partial x}(S_x^2 + S_y^2) = 0$ or $\frac{\partial}{\partial y}(S_x^2 + S_y^2) = 0$ where S_x and S_y stand for $\frac{\partial S}{\partial x}$ and $\frac{\partial S}{\partial y}$ respectively. Explain the geometrical significance of the above forms.

Solution by Hukham Chand.

Let $S \equiv (\alpha x + \beta y)^2 + 2gx + 2fy + c = 0$. We make this equation homogeneous by replacing x, y by x/z and y/z.

The point at infinity on S is $(\beta, -\alpha, 0)$ and the point at infinity on chords perpendicular to the axis is $(\alpha, \beta, 0)$. So the polars of these points, with respect to the parabola, and the polars of the circular points at infinity given by $S_x^2 + S_y^2 = 0$ form a harmonic pencil with vertex at $(\beta, -\alpha, 0)$. Thus the axis of the parabola may be obtained as the conjugate with respect to $S_x^2 + S_y^2 = 0$ of the line at infinity, and hence also as the polar of any point on the line at infinity. In particular taking the point at (1, 0, 0) or (0, 1, 0), we have the equations

$$\frac{\partial}{\partial x}(S_x^2 + S_y^2) = 0 \text{ and } \frac{\partial}{\partial y}(S_x^2 + S_y^2) = 0.$$

The geometrical significance of the above forms are brought out in the above treatment.

Also solved by S. P. Ranganathachar and A. L. Shaikh and N. K. Narasimhamurti.

Question 1653.

(K. J. SANJANA AND K. F. VAKIL) :—In the triangle ABC, each of the angles at B and C is double of the angle at A ; CM is drawn perpendicular to AB ; AM and MB are bisected at F and G respectively and CF and CG are joined.

Prove that (1) sin FCG : sin ACB = 8 : 11 ;

(2) rect. AF.BG : rect. CF.CG = 1 : 11 ;

and (3) in-radius of ΔCFG : in-radius of ΔCAB $= 5 + \sqrt{5} : 10$.

Solution by P. C. Shah, A. L. Shaikh, B. M. Narayana Rao, K. Subba Rao, M. K. Hariharan and N. K. Narasimhamurti.

Let ABC be a triangle having $\angle B = \angle C = 2\angle A$.

$\therefore$ If $AC = AB = y$ and $BC = x$, then $y^2 - x^2 = xy$;

and if $CM \perp AB$, then $AM = (y + x)/2$ and $BM = (y - x)/2$.

$\therefore$ $AF.BG = \frac{1}{2} AM.\frac{1}{2} BM = (y^2 - x^2)/16 = xy/16$.

Now $CF^2 = AC^2 - 3AF^2$ from the $\triangle ACM$

$$= y^2 - 3(y + x)^2/16 = (13y^2 - 6xy - 3x^2)/16$$

$$= (y^2 + 6xy + 9x^2)/16 \text{ [substituting for } 12y^2]$$

$$= (y + 3x)^2/16.$$

$\therefore$ $CF = (y + 3x)/4$.

Similarly $CG = (3y - x)/4$.

$\therefore$ $CF.CG = (3y^2 + 8xy - 3x^2)/16 = 11xy/16$. $\therefore \dfrac{AF.BG}{CF.CG} = \dfrac{1}{11}$

and $CF + CG + FG = (3y + x)/2$ since $FG = y/2$.

$$\therefore \quad \frac{\text{Perimeter of ABC}}{\text{Perimeter of FCG}} = \frac{2(2y + x)}{3y + x} = \frac{5 + \sqrt{5}}{5}.$$

If r, r', s, s' and Δ, Δ', be the radii, semi-perimeter and areas of Δ FCG and Δ ABC, then

$$\frac{r}{r'} = \frac{\Delta}{s} \times \frac{s'}{\Delta'} = \frac{s'}{2s} = \frac{5 + \sqrt{5}}{10} \quad \text{since } \frac{\Delta}{\Delta'} = \frac{1}{2}.$$

Again $$\frac{\Delta}{\Delta'} = \frac{\frac{1}{2} CF.CG \,.\, \sin FCG}{\frac{1}{2} AC.CB \,.\, \sin ACB} = \frac{11 \sin FCG}{16 \sin ACB} = \frac{1}{2}.$$

$$\therefore \quad \frac{\sin FCG}{\sin ACB} = \frac{8}{11}.$$

Question 1658.

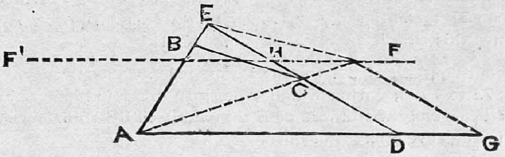
(A. Ranganatha Rao) :—Construct a quadrilateral three of whose sides are equal, given, in magnitude, the remaining side and the angles adjacent to this side.

Solution by Nathan Altshiller-Court.

Let ABCD be the required quadrilateral, AD being the side given in magnitude. The triangle ADE, where E ≡ (AB, CD), is given by its base and the adjacent angles. If through E we draw the parallel to BC meeting AC in F and through F the parallel to ED meeting AD in G, we obtain the auxiliary quadrilateral AEFG which is homothetic to ABCD, A being the homothetic center ; hence AE = EF = FG. Now the vertex F of the quadrilateral AEFG may be determined as follows. On DE lay off DH = AE, and the point F is common to the parallel through H to AD and the circle (E) having E for centre and EA for radius.

The line AF meets ED in the vertex C of the required quadrilateral ABCD, which quadrilateral is now readily completed.

The second point of intersection F′ of the line HF with the circle (E) yields a second solution of the problem. Neither of these two solutions, however, is necessarily a convex quadrilateral.



BIBLIOGRAPHICAL REFERENCES. 1. I. Alexandroff: *Problemes de geometrie elementaire*, p. 72, Hermann, Paris, 1899. Trans. from the Russian ; 2. Nathan Altshiller-Court : *College Geometry*, p. 44, Johnson Publishing Company, Richmond, Virginia ; 3. *School Science and Mathematics*, 1929, p. 644, Q. 1062.

Also solved by R. S. Vaidiswaran, N. K. Narasimhamurti, S. Srinivasan and the proposer, and by trigonometric methods by V. Sundaram, M. K. Hariharan and K. Subramanian.

REVIEWS OF BOOKS.

A History of Mathematics in America before 1900, by DAVID EUGENE SMITH, Professor Emeritus of Mathematics, Teachers' College, Columbia University, and JEKUTHIEL GINSBURG, Professor of Mathematics in Yeshiva College, New York, and Editor of "*Scripta Mathematica.*" Chicago, Illinois, 1934, pp. x + 209.

This little volume was published by the Mathematical Association of America as number 5 of the "Carus Mathematical Monographs." It is practically confined to the history of mathematics in the United States of America and hence it relates to practically the same territory as the much larger work (400 pages) by F. Cajori which was published in 1890 by the Bureau of Education of the United States under the title *The Teaching and History of Mathematics in the United States.* While the title of the former work implies that it is restricted to American Mathematics *before* 1900, this restriction is not maintained and hence this title is somewhat misleading.

About one-half of the volume is devoted to the twenty-five years from 1875 to 1900, during which American mathematicians began to co-operate actively with those of Europe, especially with those of Germany, in the development of our subject. The earlier periods are almost completely barren as regards American contributions towards the development of mathematics but they are interesting from the standpoint of the teaching of this subject in our schools and Universities. In America nearly all of the advances in mathematics were made by the professors of this subject in the colleges and the Universities. The most notable exception to this rule is G. W. Hill (1838—1914) who was connected with the Nautical Almanac of the United States for many years (1859—1892) and was later President of the American Mathematical Society (1895-1896).

Although the English language is the predominant language in the United States of America, the mathematical developments in this country since the beginning of the nineteenth century have not been predominantly influenced by English writers. During the first half of this

century the works of French mathematicians were most assiduously studied in this country and served as models for many of the text-books which were then produced therein, while in the second half of the same century those of German authors gradually b came more influential and the German Universities attracted most of the students of the United States of America who went to Europe for their advanced training in Mathematics. Since the World War, there has naturally been a great change in this respect but German mathematical literature is still very influential in this country as is evidenced by the translation into English of Felix Klein's *Elementary Mathematics from an advanced standpoint*, 1932.

The little volume under review is divided into four chapters. The third of these begins with the following sentence: "The first three centuries of our history were, as we have seen, barren of achievement in the domain of mathematics." It is somewhat difficult to harmonize this with the following quotation from page 10: "So it is not in the schools that the advance in mathematics in the sixteenth, seventeenth, and eighteenth centuries in America is to be found, but rather in the quasi-astronomical work of the astrologers and the makers of almanacs, the two classes often merging in one." On page 96 it is stated that John Farrar (1779—1853) "did much for elementary mathematics in this country through his translations 1818—1825) of the works of Euler, Lacroix, Legendre, and Bezout, and through his publication of a number of text-books."

In view of the enormous extent of these works the reader might at first be inclined to think that John Farrar was the most prolific mathematical writer that ever lived, especially since such strenuous efforts have been made in recent years to publish the collected works of L. Euler by international co-operation. These quotations are noted here because they may be interesting in themselves and may enable the reader to form a more nearly correct idea as regards the merits of the little volume under review than could otherwise be given in such a brief space. They seem to be characteristic examples of D. E. Smith's historical writings in recent years.

G. A. MILLER.

Elementary Quantum Mechanics: By Gurney, Cambridge University Press, pp. 156. Price 8*s*. 6*d*.

The purpose of the book is to explain to the physicist not acquainted with the requisite mathematical foundations, the ideas of modern quantum dynamic theory. Apart from certain indispensable mathematical assumptiouns the author does not presuppose an acquaintance with higher mathematics. The great idea of Schodinger, *viz.*—"The correlation of the quantum problem in physics to an eigen-value problem in mathematics" has been explained in a nice way by means of the idea of "fitting ψ-patterns into potential boxes." This is made clear by means of a number of examples such as the linear harmonic oscillator and the hydrogen atom. It is of course impossible thereby to comprehend fully the relation of the eigen-value problem to the modern theory; but a rough understanding of the theory and the consequent changes in the attitude of the physicist towards the allied physical events are certainly possible by studying the treatment found in the book. It would have been more valuable if the author had explained more fully how the shortcomings of the older theory have been overcome by means of these modern ideas.

The first three chapters of the book deal with the introduction of modern ideas and some simple problems. The fourth chapter deals with the uncertainty principle, the problem of many particles, the periodic table and the structure of metals. The fifth chapter is devoted to "the movement of particles" and ends with a list of select references. The later chapters of the book deal mainly with molecules. The author has attempted with a high measure of success to make the ideas clear to the non-mathematical scientist. There is also a chapter on the electronic theory of conductors and insulators from which one can obtain a rough idea of this important branch of the theory. The chapter on perturbation theory will be of considerable use to the physicist as the author has taken much pains to make the problem comprehendable without many mathematical assumptions. In the closing pages of the book descriptions of two physical events [atomic and molecular excitations and atomic collisions] are treated in detail. It is doubtful whether the mathematical appendix at the end of the book serves any useful purpose as the reader who likes to get acquainted with them can learn them much better by studying the classical treatises by Neumann Weyl, Dirac and Wan der Waaerden, etc.

Altogether, a very helpful book.

K. V. I.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

The following gentlemen have been elected members of the Society :—

Mahmad Ali Anwar, B.A. (Hon.), Dar ul-Aquil, Chahar Bagh, Jullunder

Ch. Sultan Baksh, M A., Government College, Hoshiarpur.

T. J Balwani, M.Sc , D. J. Sind College, Karachi.

Rabindranath Barna, Esq , M.Sc., B.L., Pleader, Jorhat, Assam.

G. S Daryanani, M Sc , S.T.C., Model High School, Karachi

S. P Kaushik, M Sc., Maharaja's College Bikaner

Dr. Victor Levin, Dr. Ing (Berlin). "The Retreat," Shahibag, Ahmedabad.

M. V. Vaidyanatha Sastri., Esq., M.A., Nizamiah Observatory, Begumpet (Deccan).

The Indian Physical Society which was recently formed with its head-quarters at Calcutta held its first annual meeting in January 1935. It has a membership of over 90 and besides arranging for lectures, symposiums, etc., publishes the ***Indian Journal of Physics.*** The office-bearers for 1935 are: President: Principal B. M. Sen, Calcutta; Vice-Presidents: Prof. J. B. Seth, Lahore, and Prof. Kanta Prasad, Patna; General Secretary: Prof D. M. Bose, Calcutta.

The twenty-second annual meeting of the Indian Science Congress was held at Calcutta from January 2nd to 8th, 1935, under the Presidentship of Dr. J. H. Hutton, Deputy Commissioner, Naga Hills, Assam. The sectional address for the Mathematics and Physics section was delivered by the sectional President, Dr. N. R. Sen, and dealt with "The Development of modern theoretical Physics and its limitations." The 1936 session of the Science Congres will, it is expected, meet at Indore under the Presidentship of Dr. Sir Upendranath Brahmachari.

An international Congress for Problems of Vector and Tensor analysis was held at the University of Moscow in May 1934. The Congress was organised by Prof B. Kagan and Prof. J. A. Schouten.

In 1933 the University of Madras offered a Ramanujan Memorial Prize for the best original thesis by an Indian or one domiciled in India on a definite branch of Mathematics, pure or applied. A number of theses were submitted and the

University has now announced that the prize of the value of Rs. 900 has been divided equally among the following :

S. Chandrasekharan, Esq.. Fellow, Trinity College, Cambridge ;

S Chowla, Esq., Reade, Andhra University, Waltair ;

D D. Kosambi, Esq., Professor of Mathematics, Fergusson College, Poona.

We are glad to announce that exchange relations have been recently established with the following journals :

Collected papers of the Faculty of Science, Series A, Osaka Imperial University, Osaka, Japan.

Journal of the London Mathematical Society.

Prof. C. E. Weatherburn of the University of Western Australia has been awarded the Hector Medal and Prize by the Royal Society of New Zealand, for his contributions to differential geometry.

It is with deep regret we record the sudden death in March 1925 of Dr. Ganesh Prasad, Hardinge Professor of Mathematics at the Calcutta University, President of the Calcutta Mathematical Society and Vice-President of the Benares Mathematical Society. Dr. Prasad attended a meeting of the Executive Council of the Agra University, and after making a short speech collapsed. He was removed to the Thomason Hospital where he died shortly after admission.

Dr. Prasad's connexion with the Calcutta University began in 1914 when, at the invitation of the then Vice-Chancellor, Sir Ashutosh Mookerjee, he relinquished the post of Professor of Mathematics of Queen's College, Benares, and accepted the chair of Applied Mathematics in the Calcutta University founded by Sir Rash Behari Ghosh.

The death is announced of Prof. W. Franz Meyer of the University of Knigsberg at the age of 77.

Prof. D. M Y. Somerville of Victoria College, Wellington, New Zealand, author of text-books on Analytical Geometry of 2 and of 3 dimensions and on Non-Euclidean Geometry, died in Jan. 1934 at the age of 54.

Madame Marie Curie, Professor of General Physics in the Paris University, and Director of the Laboratoire Curie at the Institute de Radium, known for her work with her husband Pierre Curie leading to the discovery of radium, died July 1934.

William de Sitter, Director of the Astronomical laboratory at Leiden and well-known for his cosmological speculations, died on Nov. 19th, 1934.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1692 (K. RANGASWAMI) :—Show that every pair of polar lines with respect to a quadric has another unique pair of polar lines intersecting the former pair, such that the two pairs meet the quadric in points the normals at which to the quadric belong to a linear complex.

Also, if the lines of the first pair are mutually perpendicular, the normals meet the line of shortest distance of this pair.

1693. (HANSRAJ GUPTA) :—If P (n, m) denote the number of partitions of n in which the least part is m, and a is any positive integer not less than 3, prove that

$$\lim \frac{P\,(am - 1,\, m)}{m^{a-3}} = \frac{1}{(a-3)!\,(a-2)!}.$$

1694. (V. THEBAULT)—France :—With the figures x, $x+1$, $x+2$, $x+3$, $x+4$ form a number whose square contains the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 each taken once.

1695. (A. A. KRISHNASWAMY AYYANGAR) :—"A coin is tossed 1000 times. Show that an absolute majority of the 2^{1000} possible sequences gives the difference between the number of heads and number of tails less than seven."

(Whittaker and Robinson : *The Calculus of observations*, p. 177).

Is this problem correct ?

1696. (K. RANGASWAMI) :—P, P′ are inverse points with respect to the polar circle of a triangle ABC. Show that the tangents at P, P′ to the rectangular hyperbolas through ABCP, ABCP′ respectively are parallel.

1697. (B. RAMAMURTI):—Given a conic and three points A_1, A_2, A_3 in the plane of the conic, show that in general there are two and only two triangles inscribed in the conic such that the sides taken in order pass through A_1, A_2 and A_3 respectively. Show that there are ∞' such triangles when and only when A_1, A_2, A_3 form the vertices of a triangle self-conjugate with respect to the conic.

Extend the result to n-gons inscribed in the conic such that the sides taken in order pass through n fixed points in the plane.

1698 (V. RAMASWAMI AIYAR):—Given four points A, B, C, D in a plane, let α, β, γ, δ be the isogonal conjugates of a point P at infinity with respect to the triangles BCD, CDA, DAB, ABC. Prove that

(i) The quadrangles $\alpha\beta\gamma\delta$, corresponding to different points P at infinity, are all similar and have a common centre of similitude

(ii) This centre of similitude is the "*Durairajan point*," M, of the quadrangle ABCD and also of each of the quadrangle $\alpha\beta\gamma\delta$.

(iii) For any quadrangle $\alpha\beta\gamma\delta$ the rectangles $MA.M\alpha$, $MB\,M\beta$, $MC.M\gamma$, $MD.M\delta$ are all equal

[As to the point M, here, called the "Durairajan point," see Mr. Durairajan's question 1480 in our Journal; and also questions 1481, 1485 and 1495 by others].

1699. (A RANGANATHA RAO):—If O is a given point in the plane of a conic S, prove that

(i) there exist four points A, B, C, D on S such that the directrix of the parabola having a contact of the third order with S at each of these points, passes through O;

(ii) A, B, C, D lie on a circle whose centre is O.

(iii) the circle ABCD is out-polar to S;

(iv) (ABCD) is the unique cyclic tetrad on S with centre O, apolar to the feet of the normals to S from O.

List of Journals and Books received in the Library of the Indian Mathematical Society between Jan. 1934 and Jan. 1935.

Abhandlungen aus dem Seminar Hamburg, **10**.

Acta Mathematica, **62**, 3, 4, **63**, 1, 2.

American Journal of Mathematics, **56**, 3, 4 (3 copies).

American Mathematical Monthly, **41**, 5 to 9 (3 copies).

Annales de l'Ecole Normale Superieure, 1, 2, 3, 1934.

Annales de l'Institute Henri Poincare, **4**, 2.

Annals of Mathematics, **35**, 2, 3, 4.

Anuario (Universidad Nacional de La Plata), No. 97.

Astro-physical Journal, **79**, 5, **80**, 1, 2, 3, 4, 5.

Bulletin des Sciences Mathematiques, 58, May to Nov. 1934.

Bolletin del Seminario Mathematica, **3**, (1932-33).

Bulletin of the American Mathematical Society, **40**, 4 to 12.

Bulletin of the Calcutta Mathematical Society, **25**, 4.

Crelle's Journal, **171**, 1, 2, 3, 4, **172**, 1, 2.

Current Science, **2**, 12, **3**, 1 to 7.

Gottinger Nachrichten, **1**, 1, 2.

Half-yearly Journal of the Mysore University, **7**, 1.

Jahresbericht der deutschen Mathematiker Vereinigung, **44**, 1 to 8.

Japanese Journal of Mathematics, **10**, 4, **11**, 1, 2.

Mathematics Teacher, **27**, 7, 8, **28**, 1.

Mathematische Annalen, **109**, 5, **110**, 1, 2, 3, 4.

Mathematical Gazette, **18**, 229, 230, 231 (3 copies).

Monthly Notices of the Royal Astronomical Society, **94**, 5 to 9, **95**, 1.

Nieuw Archief voor Wiskunde, **17**, 2.

Osaka Imperial University College, Papers, **1**, 1, 2, 3.

Philosophical Magazine, **17**, 115—120, **18**, 119—123.

Philosophical Transactions of the Royal Society of London, **233**, 721 to 730, **234**, 731, 732.

Popular Astronomy, **42**, 5 to 10.

Proceedings of the Edinburgh Mathematical Society (Index to Vols. 21—44).

Proceedings of the Cambridge Philosophical Society, **30**, 3, 4.

Proceedings of the Indian Academy of Sciences, **1**, 1.

Proceedings of the London Mathematical Society, **37**, 1 to 7, **38**, 1 to 5 (2 copies each).

Proceedings of the Physico-Mathematical Society of Japan, **16**, 5 to 12.

Proceedings of the Royal Society of London, **145**, 854, 855. **146**, 856 to 860, **147**, 861, 862, **148**, 863.

Publications de la Facultes des Sciences, 188, 190 of 1933,

Quarterly Journal of Mathematics. **5**, 18, 19, 20 (3 copies each).

Rendiconti del circlo Matematico di Palermo, **53**, 1, 2.

Rendiconti del Seminario Matematico, **12**, 1, 2.

Revue Semestrielle des publications Mathematiques (Index to Vols. **31** to **36**).

Revista Matematica Hispano Americana, **8**, 7 to 10, **9**, 1 to 9.

Science Progress, **29**, 113, 114, 115.

Scripta Mathematica. **1**, 1 to 4, **2**, 1 to 4.

Societe Mathematique de Kharoff, **8**, 9.

Sphinx, **4**, 1 to 12.

Tohoku Mathematical Journal, **39**, 1, 2.

Transactions of the Royal Society of South Africa, **21**, 4, **22**, 2, 3, 4.

Books.

Bombay University Calendar, 1930 34,

Madras University Calendar 1933-34, Vol. II, and for 1934-35 Vol. I, Parts 1 and 2.

Encyklopadie der Mathematischen Wissenschaften III_2 Heft I_2, I_3 VI_2 B Heft. 6.

Magic Cubes which are uniform step cubes, by Kirtland Mcdonald.

Mathematical Problems of Radiative Equilibrium (Camb. Tracts 31), by Eberhard Hopp.

On Solutions of Linear Partial Differential Equations near Singular places, by Kimbrell Smith.

On the Geometry of Groups of Line Configurations, by W. G. Warnock.

Planar Cremona Transformations, by S. E. Barber.

Sur deux surfaces cerclees biquadratiques, No. 191.

Sur une equation functionelle de la theorie des probabilities, No. 194.

The (r, x) system of co-ordinates and its place in Analytical Geometry and Calculus, by Collins, J. V.

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